Quiz online Due Friday

Collaboration allowed, but try alone first

↑ Cite them

9/13

Last time: Cross product

ex.: Let $\vec{u} = \langle 7, -1, 3 \rangle$, $\vec{v} = \langle -4, 9, 6 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 7 & -1 & 3 \\ -4 & 9 & 6 \end{vmatrix} = \begin{vmatrix} -1 & 3 \\ 9 & 6 \end{vmatrix} \vec{i} - \begin{vmatrix} 7 & 3 \\ -4 & 6 \end{vmatrix} \vec{j} + \begin{vmatrix} 7 & -1 \\ -4 & 9 \end{vmatrix} \vec{k}$$

$$((-1)(6) - (3)(9))\vec{i} - ((7)(6) - (3)(-4))\vec{j} + ((7)(9) - (-1)(-4))\vec{k}$$

$$(-33)\vec{i} - 54\vec{j} + 59\vec{k}$$

$$= \langle -33, -54, 59 \rangle$$

* Recall: prop (properties of the cross product)

Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ and $c \in \mathbb{R}$

1. $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
2. $(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v}) = \vec{u} \times (c\vec{v})$
3. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
4. $(\vec{u} \times \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$
5. $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$
6. $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$

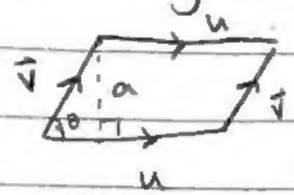
Algebraic Properties

7. $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$
8. $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$ (for the angle between $\vec{u}$ and $\vec{v}$
9. $\vec{u} \times \vec{v} = \vec{0}$ if and only if $\vec{u}$ and $\vec{v}$ are parallel

Geometric properties

Notice: Cross Product obeys "right hand rule" (direction)

As for the magnitude,



<u>Note</u> $\sin\theta = \frac{O}{H} \Rightarrow \sin\theta = \frac{a}{|\vec{v}|}$ i.e $a = |\vec{v}|\sin\theta$

$\therefore$ Area of parallelogram is

$$A = (\text{altitude})(\text{base}) = a|\vec{u}| = |\vec{u}||\vec{v}|\sin\theta$$

<u>Point</u>: If we know ⑧, we know the area of the parallelogram for $\vec{u}, \vec{v}$ is the magnitude of the cross product

<u>Proof of part ⑧ of the proposition</u>:

$$|\vec{u} \times \vec{v}|^2 = (\vec{u} \times \vec{v}) \cdot \underbrace{(\vec{u} \times \vec{v})}_{w \leftarrow \text{Apply pt. } 5} \quad \text{(property of dot product)}$$

$$= \vec{u} \cdot (\vec{v} \times (\vec{u} \times \vec{v})) \quad \text{(property ⑤)}$$

$$= \vec{u} \cdot ((\vec{v} \cdot \vec{v})\vec{u} - (\vec{v} \cdot \vec{u})\vec{v}) \quad \text{(property ⑥)}$$

$$\left.\begin{aligned} &= \vec{u} \cdot ((\vec{v} \cdot \vec{v})\vec{u}) - \vec{u} \cdot ((\vec{v} \cdot \vec{u})\vec{v}) \\ &= (\vec{v} \cdot \vec{v})(\vec{u} \cdot \vec{u}) - (\vec{v} \cdot \vec{u})(u \cdot \vec{v}) \\ &= |\vec{v}|^2 |\vec{u}|^2 - (\vec{u} \cdot \vec{v})^2 \end{aligned}\right\} \text{Properties of the Dot product}$$

$$= |\vec{u}|^2 |\vec{v}|^2 - (|\vec{u}||\vec{v}|\cos(\theta))^2 \quad \text{geometric Representation of Dot product}$$

$$= |\vec{u}|^2 |\vec{v}|^2 - |\vec{u}|^2 |\vec{v}|^2 \cos^2(\theta)$$

$$= |\vec{u}|^2 |v|^2 (1 - \cos^2\theta)$$

$$= |\vec{u}|^2 |v|^2 \sin^2\theta$$

$$= (|\vec{u}|\,|\vec{v}|\,\sin\theta)^2$$

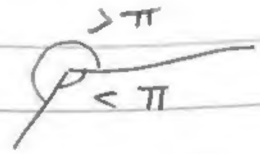
$$\therefore |\vec{u} \times \vec{v}|^2 = (|\vec{u}||\vec{v}|\sin(\theta))^2$$

$\rightarrow$

$|\vec{u} \times \vec{v}|^2 = (|\vec{u}||\vec{v}| \sin(\theta))^2$

On the other hand, θ is the geometric angle between $\vec{u}$ and $\vec{v}$

$\therefore \theta \in [0, \pi]$

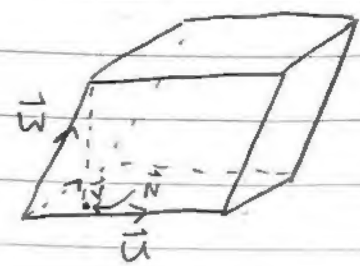


So $\sin\theta \geq 0$

Hence $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin(\theta)$ as desired :)

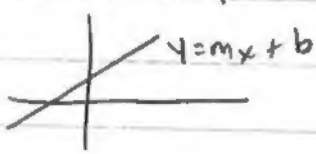
Cor: the Scalar triple product $\vec{u} \cdot (\vec{v} \times \vec{w})$ computes the signed volume of the parallelopiped one determined by $\vec{u}, \vec{v}, \vec{w}$.

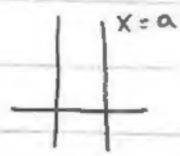
Proof is in a Video on Website



§ 12.5: Lines and planes

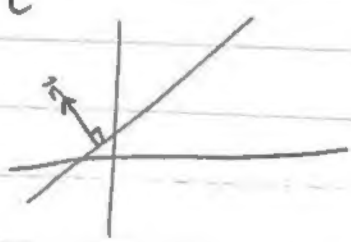
In 2-space:





$ax + by = C$ ← Better equation for a line

$\vec{n} \cdot \langle x, y \rangle = C$



In 3-space lets think about the same equation,

$$\vec{n} \cdot \vec{x} = d \qquad (\vec{n} \neq \vec{0})$$

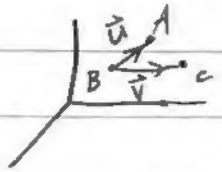
ie $\langle a,b,c \rangle \cdot \langle x,y,z \rangle = d$ Vector Equation

$ax + by + cz = d$

This is a plane in 3-space

Note: given 2 vectors (non-parallel), we get a plane. One normal vector to that plane is the cross product of the given vectors

ex: Compute an equation of the plane containing the points: $(0,1,3)$, $(2,4,0)$, and $(1,2,3)$



Sol: Note that the vectors

$$\vec{u} = \langle 2-0, 4-1, 0-3 \rangle = \langle 2,3,-3 \rangle$$

$$\vec{v} = \langle 1-0, 2-1, 3-3 \rangle = \langle 1,1,0 \rangle$$

∴ we can compute a normal vector via

$$\vec{n} = \vec{u} \times \vec{v}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -3 \\ 1 & 1 & 0 \end{vmatrix} = \langle 3, -3, -1 \rangle$$

Choose any 2 pairs of the 3 pts & they can create a normal vector

∴ the plane has equation

$\vec{n} \cdot \vec{x} = d$

i.e. $3x - 3y - z = d$

∴ using $\langle 0,1,3 \rangle$ we determine:

$d = 3 \cdot 0 - 3 \cdot 1 - 3$

$3x - 3y - z = -6$